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STABILIZATION OF TRAFFIC FLOW WITH A LEADING AUTONOMOUS VEHICLE

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ABSTRACT

This paper develops boundary control law for autonomous vehicles to stabilize the stop-and-go traffic on freeway. The macroscopic traffic dynamics is described by the Aw-Rascle-Zhang(ARZ) model in a time and state dependent domain. The leading autonomous vehicle aims to regulate the traffic behind it to uniform equilibrium and the domain length of the traffic to a setpoint. The traffic density and speed is governed by second-order, nonlinear hyperbolic partial differential equations(PDEs), coupled with a state-dependent ODE for the leading autonomous vehicle. The actuation is the speed of autonomous vehicle at the moving front boundary of the domain. We linearize the system around a uniform velocity and density reference and certain physical properties are discussed for the model validity. The linearized model describes the dynamics of deviations of density and velocity from the reference. By transforming the linearized system in a moving coordinate, we obtain a domain with a fixed boundary at one end and a state-dependent moving boundary at the other end. The well-posedness of the system is proved and the linear instability of open-loop system is shown. We further map the system to Riemann variables and based on it, propose the boundary feedback control law actuated by the leading autonomous vehicle. The exponential stability of state variables in L_2 norm and convergence to the setpoint domain length is achieved for the closed-loop system.

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INTRODUCTION

Traffic flow congestion on unidirectional freeways has been investigated intensively during the past years. The motivations behind are to understand the formation of traffic congestion, and suppress the instabilities of traffic flow.

Macroscopic modeling of traffic dynamics using PDEs has been proposed, including first-order model by Ligthill and Whitham and Richards, second-order Payne-Whitham model and second-order Aw-Rascle Zhang(ARZ) model [1] [2]. The stop-and-go traffic [3] [4] [6] [7], also known as "jamiton", can rise without apparent inhomogeneities of drivers and the road situations. It has been observed as a common phenomenon in traffic and demonstrated by experiments [5]. The phantom traffic jam has been widely studied with ARZ model [6] [7] [9], which consists of second-order, nonlinear hyperbolic PDEs of traffic density and velocity. The theoretical linear stability analysis on uniform steady states of the nonlinear hyperbolic ARZ model can be performed. The instabilities appear in the congested regime of the ARZ model [10] [11], as a result of the collective behavior of drivers. The equilibrium uniform vehicle velocity and density is unstable and develops into traffic instabilities.

Boundary control strategies through ramp metering and varying speed limit have proven to be very effective in freeway traffic management. Boundary control of traffic flow aims to stabilize the oscillations of traffic density and velocity which cause increase consumptions of fuel and unsafe driving conditions. Many recent efforts [9] [8] [12] [13] are focused on the boundary control of ARZ model. [9] applied spectral analysis to the linearized ARZ model, upon which the boundary control and measurement are designed. [12] investigated the local stability of a positive hyperbolic system with application to the ARZ model and [13] provided a boundary control law that achieved global stabilization. [10] [14] firstly applied the backstepping boundary control design to stabilize stop-and-go traffic of ARZ model with ramp metering.

The previous cited results dealt with the fixed boundary control of traffic flow. The coordinate system is fixed to the ground while the traffic flow travels downstream. By the time traffic flow in the fixed domain is stabilized, the initial oscillations generating vehicles have left the domain. The problem arises when the regulation of traffic flow is focused on the moving vehicles. For example, there is traffic congestion caused by accidents or work zones in the downstream of traffic flow. We aim to regulate the vehicles to a uniform desired velocity before entering the accident zone. The fixed boundary control fails here but autonomous vehicles open new possibilities. On one hand, autonomous vehicles have great advantages in accessing to non-local information, compared with ramp metering and varying speed limit. On the other hand, autonomous vehicles excel in following the designed driving rules precisely and rapidly, compared with human drivers.

This idea has been developed for a one-lane ring road with actuation of a single autonomous vehicle in [15]. Based on car-following model, autonomous vehicle locally stabilize the traffic flow to achieve string stability on the ring road. This result is limited by the nature of car-following model. Furthermore, [16] considers the macroscopic model and use autonomous vehicles as moving bottlenecks to reduce the overall fuel consumptions. The macroscopic model with moving bottlenecks is governed by a coupled PDE-ODE model [17] [18]. The traffic dynamics is governed by PDEs and the moving boundary is governed by an ODE. The dynamics of the moving bottleneck depends on the physical properties of the slow-moving vehicles and their drivers. In this context, instead of being impeded by slow-moving vehicles, we control the speed of the moving boundary, aiming to stabilize the traffic flow behind the leading autonomous vehicle.

The main contribution of this paper is to establish the first result on stabilization of ARZ traffic model with a leading autonomous vehicle. The mathematical model is a nonlinear hyperbolic PDE system coupled with state-dependent boundary input ODE in a moving coordinate. While the boundary control design for nonlinear PDE-ODE systems with a moving boundary has been studied intensively in [19] [20] for hyperbolic PDEs and in [21] for a parabolic PDE, their systems have the instability source not on PDEs but on the ODE. On the contrary, the system we consider in this paper has unstable effects on PDEs, and we propose a novel control design using coordinate transformation and Lyapunov analysis. We also prove that the control design guarantees that some physical conditions on the model validity are met.

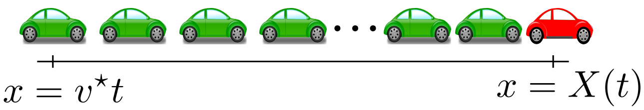

FIGURE 1. TRAFFIC MODEL WITH AUTONOMOUS VEHICLE AS LEADING CAR

The outline is as follows: we first introduce the macroscopic ARZ traffic model in a time and state-dependent, one-dimensional spatial domain. The autonomous vehicle serves as moving front boundary and actuate the model through the velocity. Then we linearize the nonlinear coupled hyperbolic PDE-ODE system around uniform steady states. We consider the problem relative to a moving coordinate and proved the well-posedness of the linearized system in the moving coordinate. The linear stability analysis is applied for the open-loop system. We design a boundary control law that stabilizes the system to uniform steady states and the desired domain setpoint. Lyapunov analysis guarantees the exponential stability of the state variables in L_2 norm. It is also proved that the control law design does not violate the model validity. We end our paper with stating the conclusion and future work.

PROBLEM STATEMENT

The macroscopic traffic dynamics is governed by the ARZ model on one dimensional spatial coordinate x, which consists of second-order, nonlinear hyperbolic PDEs of traffic density and velocity denoted as $\rho(x,t)$ and $v(x,t)$, respectively. To reduce the stop-and-go phenomenon, in this paper we utilize an autonomous vehicle located at $x = X(t)$ as a moving boundary so that the traffic density and velocity profile behind is driven to desired uniform density $\rho^\star$ and velocity $v^\star$. The diagram is shown in Fig.1.

Since we focus on stabilization of the traffic flow behind the automatic vehicles to desired velocity $v^\star$, the domain we consider is a time-varying spatial domain given by

$$\forall (x,t) \in [v^\star t, X(t)] \times [0,\infty). \tag{1}$$

The ARZ model describes the second-order traffic PDE in $(\rho(x,t), v(x,t))$ as

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{2}$$

$$\partial_t v + (v - \rho p'(\rho))\partial_x v = \frac{V(\rho) - v}{\tau}, \tag{3}$$

where τ is the relaxation time that relates to driving behavior of drivers adapting to the equilibrium velocity. The variable $p(\rho)$

denotes the traffic pressure, which is an increasing function of the density given by

$$p(\rho) = c_o (\rho)^{\gamma}, \tag{4}$$

with $\gamma \in \mathbb{R}_+$. The equilibrium velocity-density relationship $V(\rho)$ is given in Greenshield model,

$$V(\rho) = v_f \left(1 - \frac{\rho}{\rho_m}\right), \tag{5}$$

where v_f is the free flow velocity, ρ_m is the maximum density.

BOUNDARY CONTROL MODEL

The position of autonomous vehicles $X(t)$ is governed by the following ODE

$$\dot{X}(t) = U(t) + v^\star, \tag{6}$$
$$X(0) = L, \tag{7}$$

where $U(t)$ is the velocity actuation relative to the setpoint velocity $v^\star$ that we design. The velocity variation control input is realized with an autonomous vehicle at the leading position. The positive constant L is defined as the length of freeway segment at the initial time $t = 0$. Taking integration of (6) in time with the initial condition (7), the explicit expression of $X(t)$ is given by

$$X(t) = \int_0^t U(\mu) d\mu + v^\star t + L. \tag{8}$$

Finally, the boundary conditions of the PDEs are given by

$$\rho(v^\star t, t) v(v^\star t, t) = q^\star, \tag{9}$$
$$v(X(t), t) = U(t) + v^\star. \tag{10}$$

The assumption is that we keep a constant traffic flux $q^\star = \rho^\star v^\star$ at $x = v^\star t$. The other moving boundary condition is actuated with the velocity of the leading autonomous vehicle.

Physical Properties

For vehicles driving on the freeway, some physical properties of the model are stated for the model validation.

Remark 1. *The velocity of the leading autonomous vehicle in* (10) *is positive relative to the fixed spatial coordinate. Therefore, the control law design needs to satisfy* $\forall t > 0$

$$U(t) > -v^\star. \tag{11}$$

Remark 2. *The domain length of the model is state-dependent and must maintain positive. Hence, the control law design needs to satisfy* $\forall t > 0$

$$X(t) - v^\star t = \int_0^t U(\mu) d\mu + L > 0. \tag{12}$$

The physical properties of the model need to be satisfied with the proposed control law later. This will be further addressed in the following section.

Linearized ARZ Around Uniform Reference

The ARZ model in (2), (3) with boundary conditions in (9) (10) is a nonlinear second-order hyperbolic PDEs with two moving boundary conditions. We linearize the ARZ model (2) and (3) around the steady states ($\rho^\star$, $v^\star$) with $v^\star = V(\rho^\star)$ given in (5). Let $(\tilde{\rho}, \tilde{v})$ be the small deviations from the nominal profile defined by

$$\tilde{\rho}(x,t) := \rho(x,t) - \rho^\star, \quad \tilde{v}(x,t) := v(x,t) - v^\star. \tag{13}$$

Then, the linearized PDE model for $(\tilde{\rho}, \tilde{v})$ is described by

$$\partial_t \tilde{\rho} + v^\star \partial_x \tilde{\rho} = -\rho^\star \partial_x \tilde{v}, \tag{14}$$
$$\partial_t \tilde{v} - (\rho^\star p'(\rho^\star) - v^\star) \partial_x \tilde{v} = \frac{\tilde{\rho} V'(\rho^\star) - \tilde{v}}{\tau}, \tag{15}$$

with the boundary conditions

$$\tilde{\rho}(v^\star t, t) = -\frac{\rho^\star}{v^\star} \tilde{v}(v^\star t, t), \tag{16}$$
$$\tilde{v}(X(t), t) = U(t). \tag{17}$$

The actuated moving boundary is governed by the following ODE,

$$\dot{X}(t) = U(t) + v^\star, \tag{18}$$
$$X(0) = L. \tag{19}$$

We consider the congested regime for this problem in which $\tilde{\rho}$ transports downstream and $\tilde{v}$ transports upstream which imposes the following relation

$$\rho^\star p'(\rho^\star) = \gamma p^\star > v^\star. \tag{20}$$

Linearized ARZ in Moving Coordinate

We are dealing with a hereto-directional second-order PDE and ODE coupled system. The inlet boundary is time-varying and the actuated boundary is state-dependent.

Since both boundary conditions are moving, we define a new spatial coordinate so that one boundary is fixed. The new coordinate is defined as

$$\bar{x} = x - v^\star t. \tag{21}$$

The domain of the new coordinate is denoted as

$$\bar{x} \in [0, \bar{X}(t)], \tag{22}$$

where the actuated moving boundary is

$$\bar{X}(t) = X(t) - v^\star t = \int_0^t U(\mu) d\mu + L, \tag{23}$$

$$\bar{X}(0) = L. \tag{24}$$

Then we find that for state variables $\alpha := [\tilde{\rho}, \tilde{v}]$ in new coordinates, the following relations through the differentiation are given

$$\begin{aligned}\frac{\partial \alpha}{\partial t} &= \frac{\partial \alpha}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial \alpha}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial t} \\ &= \frac{\partial \alpha}{\partial t} - v^\star \frac{\partial \alpha}{\partial \bar{x}},\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{\partial \alpha}{\partial x} &= \frac{\partial \alpha}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} + \frac{\partial \alpha}{\partial t}\frac{\partial t}{\partial x} \\ &= \frac{\partial \alpha}{\partial \bar{x}},\end{aligned} \tag{26}$$

Using the relations (25) and (26), the linearized ARZ model in (14)-(19) is rewritten in the new coordinate

$$\partial_t \tilde{\rho}(x,t) = -\rho^\star \partial_x \tilde{v}(x,t), \tag{27}$$

$$\partial_t \tilde{v}(x,t) - \rho^\star p'(\rho^\star)\partial_x \tilde{v}(x,t) = \frac{\tilde{\rho} V'(\rho^\star) - \tilde{v}(x,t)}{\tau}, \tag{28}$$

$$\tilde{\rho}(0,t) = -\frac{\rho^\star}{v^\star}\tilde{v}(0,t), \tag{29}$$

$$\tilde{v}(\bar{X}(t),t) = U(t), \tag{30}$$

where the actuated moving boundary $\bar{X}(t)$ is governed by an ODE,

$$\dot{\bar{X}}(t) = U(t), \tag{31}$$

$$\bar{X}(0) = L. \tag{32}$$

Here we use x instead of $\bar{x}$ for simplicity of expression. We will use this model in the moving coordinate for control design.

Well-posedness of Linearized Model

To show the well-posedness of the above linearized model with moving boundary, we introduce the scaling of the spatial domain $\xi \in [0,1]$, defined by

$$\xi = \frac{\bar{x}}{\bar{X}(t)}. \tag{33}$$

Following (25), (26), we have

$$\frac{\partial \alpha}{\partial t} = \frac{\partial \alpha}{\partial t} - \frac{\xi \dot{X}(t)}{\bar{X}(t)}\frac{\partial \alpha}{\partial \xi}, \tag{34}$$

$$\frac{\partial \alpha}{\partial x} = \frac{1}{\bar{X}(t)}\frac{\partial \alpha}{\partial \xi}. \tag{35}$$

Applying the relations (34), (35) to the system (27)–(32), we obtain the fixed domain model for $\xi \in [0,1]$ as

$$\partial_t \tilde{\rho}(\xi,t) = \frac{\xi \dot{X}(t)}{\bar{X}(t)}\partial_\xi \tilde{\rho}(\xi,t) - \frac{\rho^\star}{\bar{X}(t)}\tilde{v}(\xi,t), \tag{36}$$

$$\begin{aligned}\tilde{v}_t(\xi,t) = &\frac{(\gamma p^\star + \xi \dot{\bar{X}}(t))}{\bar{X}(t)}\tilde{v}_\xi(\xi,t) \\ &+ \frac{\tilde{\rho}(\xi,t)V'(\rho^\star) - \tilde{v}(\xi,t)}{\tau}\end{aligned} \tag{37}$$

$$\tilde{\rho}(0,t) = -\frac{\rho^\star}{v^\star}\tilde{v}(0,t), \tag{38}$$

$$\tilde{v}(1,t) = U(t), \tag{39}$$

$$\dot{\bar{X}}(t) = U(t). \tag{40}$$

For the above system to be well-posed, the transport velocity of the hyperbolic equation should have the correct sign. Thus, we need

$$\frac{(\gamma p^\star + \xi \dot{\bar{X}}(t))}{\bar{X}(t)} > 0. \tag{41}$$

The positivity of $\bar{X}(t)$ will be proved after we propose the control law. On the condition $\bar{X}(t) > 0$, we apply (40). The above well-posedness condition requires for $\xi \in [0,1]$

$$\gamma p^\star + \xi U(t) > 0, \tag{42}$$

which imposes a constraint for control law design. We state it in Remark 3.

Remark 3. *For the well-posedness of the linearized ARZ model, the following inequality of control law needs to be satisfied for* $t > 0$

$$U(t) > -\gamma p^\star. \tag{43}$$

If the Remark 1 is satisfied, the congested traffic condition (20) ensures that Remark 3 is met. Therefore, the well-posedness of the linearized system is guaranteed with the control law designed satisfying Remark 1.

Stability Analysis of Open-loop System

We consider the open-loop system with zero input. Thus the autonomous vehicle leads the traffic flow with a constant steady velocity $v^\star$. In the moving coordinate, the open-loop system is given for $x \in [0,L]$,

$$\partial_t \tilde{\rho}(x,t) = -\rho^\star \partial_x \tilde{v}(x,t), \tag{44}$$

$$\partial_t \tilde{v}(x,t) - \rho^\star p'(\rho^\star)\partial_x \tilde{v}(x,t) = \frac{\tilde{\rho}(x,t)V'(\rho^\star) - \tilde{v}(x,t)}{\tau}, \tag{45}$$

$$\tilde{\rho}(0,t) = -\frac{\rho^\star}{v^\star}\tilde{v}(0,t), \tag{46}$$

$$\tilde{v}(L,t) = 0, \tag{47}$$

Applying Fourier transform in x and Laplace transform in t, the following solution is considered

$$\tilde{\rho}(x,t) = \rho(k)\exp\left(\frac{\mathrm{i}kx}{L} + \lambda(k)t\right), \tag{48}$$

$$\tilde{v}(x,t) = v(k)\exp\left(\frac{\mathrm{i}kx}{L} + \lambda(k)t\right). \tag{49}$$

Substituting into (44)-(47), we obtain

$$\begin{pmatrix} \lambda & \frac{\mathrm{i}k\rho^\star}{L} \\ -\frac{V'(\rho^\star)}{\tau} & \lambda - \frac{\mathrm{i}k\rho^\star p'(\rho^\star)}{L} + \frac{1}{\tau} \end{pmatrix} \begin{pmatrix} \rho(k) \\ v(k) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

To obtain the kth pair of eigenvalues, we need to solve the quadratic equation,

$$0 = \lambda^2 - \left(\frac{\rho^\star p'(\rho^\star)}{L}\mathrm{i}k - \frac{1}{\tau}\right)\lambda + \frac{\rho^\star V'(\rho^\star)}{\tau L}\mathrm{i}k.$$

The discriminant is given by

$$\Delta = \left(\frac{\rho^\star p'(\rho^\star)}{L}\mathrm{i}k\right)^2 - \frac{2\rho^\star(2V'(\rho^\star) + p'(\rho^\star))}{\tau L}\mathrm{i}k + \left(\frac{1}{\tau}\right)^2. \tag{50}$$

When we consider

$$p'(\rho^\star) = -V'(\rho^\star) \implies \Delta = \left(\frac{\rho^\star p'(\rho^\star)}{L}\mathrm{i}k + \frac{1}{\tau}\right)^2, \tag{51}$$

there are two sets of eigenvalues

$$\lambda_1 = \frac{\rho^\star p'(\rho^\star)}{L}\mathrm{i}k, \tag{52}$$

$$\lambda_2 = -\frac{1}{\tau}. \tag{53}$$

The real parts of two eigenvalues are nonnegative. The open-loop system $(\tilde{\rho}, \tilde{v})$ over $(\rho^\star, v^\star)$ is unstable when

$$p'(\rho^\star) < -V'(\rho^\star) \implies \Delta > \left(\frac{\rho^\star p'(\rho^\star)}{L}\mathrm{i}k + \frac{1}{\tau}\right)^2, \tag{54}$$

and thus

$$\mathrm{Re}\left(\sqrt{\Delta}\right) > \frac{1}{\tau}. \tag{55}$$

The positive real part appears in λ_1,

$$\mathrm{Re}(\lambda_1) = \frac{1}{2}\left(-\frac{1}{\tau} + \mathrm{Re}\left(\sqrt{\Delta}\right)\right) > \frac{1}{2}\left(-\frac{1}{\tau} + \frac{1}{\tau}\right) = 0. \tag{56}$$

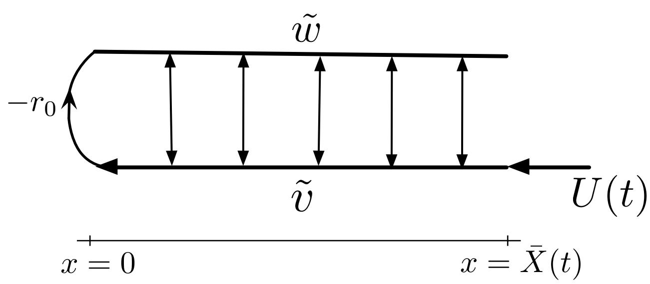

FIGURE 2. SCHEMATICS OF PDE-ODE SYSTEM

From the above stability analysis, we can conclude that the open-loop system is unstable under the condition that

$$p'(\rho^\star) < -V'(\rho^\star). \tag{57}$$

Thus the following inequality for parameters in the traffic model stands as a condition for the open-loop unstable

$$\frac{\gamma p^\star}{\rho^\star} < \frac{v_f}{\rho_m}. \tag{58}$$

Hereafter, we assume that (58) holds and use it later for the system parameter analysis. In the following sections, we derive the control design to stabilize the system.

BOUNDARY CONTROL DESIGN

This section provides the design of control law with stating our main result. First, we introduce a Riemann variable and introduce transformation to cancel the source term of instability. Next, we show that the closed-loop system under the designed controller ensures the physical properties for the model validation. Finally, we state our main result with proving the exponential stability of the closed-loop system via Lyapunov analysis.

Mapping to Riemann Variables

The cross term of the spatial derivative appearing in (44) makes analysis of the system and control design elusive. To reduce the complexity, we introduce a Riemann variable defined by

$$\tilde{w}(x,t) = \frac{\gamma p^\star}{\rho^\star}\tilde{\rho}(x,t) + \tilde{v}(x,t), \tag{59}$$

and then map the $(\tilde{\rho}, \tilde{v})$-system in (27)-(32) to $(\tilde{w}, \tilde{v})$-system. The schematics of the coupled PDE-ODE modelis shown in fig.2.

The second-order PDEs degenerate to an ODE and a first-order PDE, given by

$$\partial_t \tilde{v}(x,t) - \gamma p^\star \partial_x \tilde{v}(x,t) = - c_1 \tilde{w}(x,t) + c_2 \tilde{v}(x,t), \tag{60}$$

$$\tilde{v}(\bar{X}(t),t) = U(t), \tag{61}$$

$$\partial_t \tilde{w}(x,t) = - c_1 \tilde{w}(x,t) + c_2 \tilde{v}(x,t), \tag{62}$$

$$\tilde{w}(0,t) = - r_0 \tilde{v}(0,t), \tag{63}$$

where the actuated moving boundary $\bar{X}(t)$ is governed by an ODE,

$$\dot{\bar{X}}(t) = U(t). \tag{64}$$

The constants are defined as

$$c_1 = \frac{1}{\tau}\frac{v_f}{\rho_m}\frac{\rho^\star}{\gamma p^\star}, \tag{65}$$

$$c_2 = \frac{1}{\tau}\left(\frac{v_f}{\rho_m}\frac{\rho^\star}{\gamma p^\star} - 1\right), \tag{66}$$

$$r_0 = \frac{\gamma p^\star - v^\star}{v^\star}, \tag{67}$$

Due to the condition (58), the constants satisfy the followings

$$c_1 > \frac{1}{\tau} > 0, \tag{68}$$

$$c_2 = c_1 - \frac{1}{\tau} > 0. \tag{69}$$

Boundary Control Law

We scale the PDE state $\tilde{v}(x,t)$ to cancel the diagonal source term on the right hand side in (60). Let $\beta(x,t)$ be a state defined by

$$\beta(x,t) = \exp\left(\frac{c_2}{\gamma p^\star}x\right)\tilde{v}(x,t) \tag{70}$$

Taking the time and spatial derivative of (70) along with (60), we have a first-order hyperbolic PDE with in domain spatially varying coupling.

$$\partial_t \beta(x,t) - \gamma p^\star \partial_x \beta(x,t) = \bar{c}_2(x)\tilde{w}(x,t), \tag{71}$$

$$\beta(\bar{X}(t),t) = r(t)U(t), \tag{72}$$

with ODE

$$\partial_t \tilde{w}(x,t) = -c_1 \tilde{w}(x,t) + c_2 \exp\left(-\frac{c_2}{\gamma p^\star} x\right) \beta(x,t), \quad (73)$$

$$\tilde{w}(0,t) = -r_0 \beta(0,t), \quad (74)$$

The spatially-varying in domain coefficient is defined as

$$\bar{c}_2(x) = -c_1 \exp\left(\frac{c_2}{\gamma p^\star} x\right) < 0, \quad (75)$$

and the new spatially-varying boundary coefficient is

$$r(t) = \exp\left(\frac{c_2}{\gamma p^\star} \bar{X}(t)\right). \quad (76)$$

Let $X^\star$ be a desired domain length as one of the control objectives, and we define an ODE state $Y(t)$,

$$Y(t) = \bar{X}(t) - X^\star, \quad (77)$$

$$\dot{Y}(t) = U(t), \quad (78)$$

where the desired domain length $X^\star > 0$. The positivity of $Y(t)$ and $\bar{X}(t)$ will be proved later.

The target system is given by

$$\partial_t \beta(x,t) = \gamma p^\star \partial_x \beta(x,t) + \bar{c}_2(x) \tilde{w}, \quad (79)$$

$$\beta(\bar{X}(t),t) = -\frac{Y(t)}{T}, \quad (80)$$

$$\dot{Y}(t) = U(t), \quad (81)$$

$$\partial_t \tilde{w}(x,t) = -c_1 \tilde{w}(x,t) + c_2 \exp\left(-\frac{c_2}{\gamma p^\star} x\right) \beta(x,t), \quad (82)$$

with the control law

$$U(t) = -\frac{Y(t)}{T} \exp\left(-\frac{c_2}{\gamma p^\star} \bar{X}(t)\right), \quad (83)$$

where the positive constant $\frac{1}{T} > 0$ is the control gain based on the choice of time constant T. Under the control law (83), we have the following property.

Lemma 1. *If $Y(0) = L - X^\star > 0$, then $\forall t \geq 0$*

$$0 < Y(t) \leq L - X^\star, \quad (84)$$

$$X^\star < \bar{X}(t) \leq L. \quad (85)$$

Proof. Substituting the control law (83) into (81), we have

$$\dot{Y}(t) = -\frac{Y(t)}{T} \exp\left(-\frac{c_2}{\gamma p^\star}(Y(t) + X^\star)\right)$$
$$= -\varepsilon Y(t) \exp(-AY(t)), \quad (86)$$

where we defined $A = \frac{c_2}{\gamma p^\star} > 0$ and

$$\varepsilon = \frac{\exp(-AX^\star)}{T} > 0. \quad (87)$$

Let $Z(t) = AY(t)$. Then, we have the differential equation for $Z(t)$ as

$$\dot{Z}(t) = -\varepsilon Z(t) \exp(-Z(t)). \quad (88)$$

If $Z(0) > 0$, the solution to (88) satisfies $0 < Z(t) \leq Z(0)$ for all $t \geq 0$. Hence, it holds that if $Y(0) > 0$ then $0 < Y(t) \leq Y(0) = L - X^\star$ for all $t \geq 0$. Since $\bar{X}(t) = Y(t) + X^\star$, we also have $X^\star < \bar{X}(t) \leq L$. □

In Lemma 1, we establish the positivity of $Y(t)$ and $\bar{X}(t)$ with the control law given in (83). In practice, the autonomous vehicle starts with traffic flow of domain length L behind it, follow the control law and finally are able to stabilize traffic flow of domain length $X^\star$ behind it. In the following Lemma, we state the property of setpoint domain length.

Lemma 2. *If the control gain $T > 0$ is chosen to satisfy the following inequality,*

$$T > \frac{\exp\left(-(\frac{c_2}{\gamma p^\star} X^\star + 1)\right)}{v^\star} \frac{\gamma p^\star}{c_2}, \quad (89)$$

then $-v^\star < U(t) < 0$, which ensures that the model validity in Remark 1 holds.

Proof. By (83), we can rewrite the control law as

$$U(t) = -AY(t) \exp(-BY(t)) \quad (90)$$

where $A = \frac{\exp\left(-\frac{c_2}{\gamma p^\star} \bar{X}^\star\right)}{T} > 0$ and $B = \frac{c_2}{\gamma p^\star} > 0$. Let $f(x) = -Ax\exp(-Bx)$. Then, obviously $f(x) < 0$ for all $x \geq 0$, and $f(x) \to 0$ as $x \to \infty$. Also, we have $f'(x) = -A(1 -$

$Bx)\exp(-Bx)$, and thus $f'(x)=0$ only when $x=\frac{1}{B}$. Hence, the minimum value of $f(x)$ on $x\geq 0$ is $\inf_{x\in[0,\infty]} f(x)=f(\frac{1}{B})=-\frac{A}{B}\exp(-1)$. Taking back to (90), due to the positivity of $Y(t)$ proved in Lemma 1, we can deduce that $-\frac{A}{B}\exp(-1)<U(t)<0$. Hence, finally we have

$$-\frac{\exp\left(-(\frac{c_2}{\gamma p^\star}\bar{X}^\star+1)\right)}{T}\frac{\gamma p^\star}{c_2}<U(t)<0. \tag{91}$$

For $U(t)>-v^\star$ to hold, we impose

$$\frac{\exp\left(-(\frac{c_2}{\gamma p^\star}X^\star+1)\right)}{T}\frac{\gamma p^\star}{c_2}<v^\star, \tag{92}$$

which concludes Lemma 3.

□

Note that setpoint $X^\star$ represents the final domain length that the controller can achieve. It reflects the capability of control law under the prescribed parameters and control gain. Next we state our main result in the following theorem:

Theorem 1. *Consider a closed-loop system in* $\forall(x,t)\in[v^\star t, X(t)]\times[0,\infty)$ *consisting of the plant* (14)-(17) *with moving boundary* (18)-(19)*, and the control law*

$$U(t)=-\frac{\bar{X}(t)-X^\star}{T}\exp\left(-\frac{c_2}{\gamma p^\star}\bar{X}(t)\right), \tag{93}$$

where $\bar{X}(t)=X(t)-v^\star t$*.The initial conditions are* $\tilde{\rho}_0,\tilde{v}_0\in L^2[0,L]$*. Assume that the initial length* $L>0$ *is chosen to satisfy*

$$L<\frac{\gamma p^*}{c_2}. \tag{94}$$

Then for any reference system $(\rho^\star,v^\star,X^\star)$ *that satisfy* (58)*, the closed-loop system is exponentially stable in the sense of* L_2 *norm*

$$\left(\int_0^{\bar{X}(t)}(\rho(x,t)-\rho^\star)^2dx+\int_0^{\bar{X}(t)}(v(x,t)-v^\star)^2dx +(\bar{X}(t)-X^\star)^2\right)^{\frac{1}{2}} \tag{95}$$

Stability Analysis

In this section, we establish the exponential stability of the closed-loop system in L_2 norm from the Lyapunov analysis of the target system in $(\beta(x,t),\tilde{w}(x,t),Y(t))$. We construct the following Lyapunov function

$$V(t)=\frac{a}{2}\int_0^{\bar{X}(t)}\tilde{w}^2(x,t)dx+\frac{1}{2}\int_0^{\bar{X}(t)}(1+bx)\beta^2(x,t)dx +\frac{p}{2}Y^2(t), \tag{96}$$

where the coefficients a,b,p are positive constants. Then by taking the time derivative along the solution of target system in (79)-(82), we have

$$\begin{aligned}\dot{V}=&-ac_1\int_0^{\bar{X}(t)}\tilde{w}^2(x,t)dx\\&+\int_0^{\bar{X}(t)}\theta(x)\beta(x,t)\tilde{w}(x,t)dx\\&+\frac{a}{2}U(t)\tilde{w}^2(\bar{X}(t),t)\\&-\frac{\gamma p^\star b}{2}\int_0^{\bar{X}(t)}\beta^2(x,t)dx\\&+\frac{\gamma p^\star(1+b\bar{X}(t))}{2}\beta^2(\bar{X}(t),t)-\frac{\gamma p^\star}{2}\beta^2(0,t)\\&+\frac{1+b\bar{X}(t)}{2}U(t)\beta^2(\bar{X}(t),t)\\&-\frac{p}{T}\exp\left(-\frac{c_2}{\gamma p^\star}\bar{X}(t)\right)Y^2(t),\end{aligned} \tag{97}$$

where

$$\theta(x)=ac_2\exp\left(-\frac{c_2}{\gamma p^\star}x\right)-c_1(1+bx)\exp\left(\frac{c_2}{\gamma p^\star}x\right). \tag{98}$$

Using Cachy-Schwarz and Young's inequality, we obtain the following inequality

$$\begin{aligned}&\int_0^{\bar{X}(t)}\theta(x)\beta(x,t)\tilde{w}(x,t)dx\\&\leq\frac{\Theta d_1}{2}\int_0^{\bar{X}(t)}\beta^2(x,t)dx+\frac{1}{2d_1}\int_0^{\bar{X}(t)}\tilde{w}^2(x,t)dx\end{aligned} \tag{99}$$

where $d_1 > 0$ is a positive constant to be determined later, and $\Theta > 0$ is an upper bound of the function $\theta(x)^2$ defined as

$$\Theta := \sup_{x\in[0,L]} \theta(x)^2, \tag{100}$$

with the help of (85). By Lemma 1, the followings hold

$$\frac{a}{2}U(t)\tilde{w}^2(\bar{X}(t),t) <0, \tag{101}$$

$$\frac{1+b\bar{X}(t)}{2}U(t)\beta^2(\bar{X}(t),t) <0. \tag{102}$$

Also, by the boundary condition (72) with the control law (83), we have

$$\frac{\gamma p^\star(1+b\bar{X}(t))}{2}\beta^2(\bar{X}(t),t) \leq \frac{\gamma p^\star(1+bL)}{2}\frac{Y(t)^2}{T^2}. \tag{103}$$

Applying (99)–(103) into (97), we have

$$\begin{aligned}\dot{V} \leq &-\left(ac_1 - \frac{1}{2d_1}\right)\int_0^{\bar{X}(t)} \tilde{w}^2(x,t)dx \\ &- \frac{\gamma p^\star b - \Theta d_1}{2}\int_0^{\bar{X}(t)} \beta^2(x,t)dx \\ &- \frac{1}{T}\left(p\exp\left(-\frac{c_2L}{\gamma p^\star}\right) - \frac{\gamma p^\star(1+bL)}{2T}\right)Y^2(t).\end{aligned}$$

We need to choose a,b,p,d_1 so that the followings hold

$$ac_1 - \frac{1}{2d_1} >0, \tag{104}$$

$$\frac{\gamma p^\star b - \Theta d_1}{2} >0, \tag{105}$$

$$p > \frac{\gamma p^\star(1+bL)}{2T}\exp\left(\frac{c_2L}{\gamma p^\star}\right). \tag{106}$$

For the existence of $d_1 > 0$ such that both (104) and (105) hold, the following condition is required

$$\Theta < 2ac_1\gamma p^\star b. \tag{107}$$

Choose $a = \frac{kc_1}{c_2}$ with $k > 1$. Then, by definition of $\theta(x)$ in (98), we can see $\theta(0) = (k-1)c_1 > 0$, and $\theta(x)$ is a monotonically decreasing function in x with $\theta(x) \to -\infty$ as $x \to \infty$. Therefore, the maximum value of $\theta(x)^2$ on $x \in [0,L]$ is either $\theta(0)^2$ or $\theta(L)^2$. By imposing the condition $\theta(0) \geq -\theta(L)$ which yields

$$k \geq \frac{1+(1+bL)e^{\frac{c_2}{\gamma p^\star}L}}{1+e^{-\frac{c_2}{\gamma p^\star}L}}, \tag{108}$$

then it holds

$$\Theta = (k-1)^2c_1^2. \tag{109}$$

Also, substituting (109) and $a = \frac{kc_1}{c_2}$ into (107), we need $b > \frac{c_2}{2\gamma p^\star}(k-2+\frac{1}{k})$. Combining this and the condition (108), we have

$$\left(1+e^{-\bar{C}} - \frac{\bar{C}}{2}e^{\bar{C}}\right)k^2 - \left(1+e^{\bar{C}} - \bar{C}e^{\bar{C}}\right)k - \frac{\bar{C}}{2}e^{\bar{C}} \geq 0, \tag{110}$$

where $\bar{C} = \frac{c_2}{\gamma p^*}L$. If $\bar{C} < 1$, then $1+e^{-\bar{C}} - \frac{\bar{C}}{2}e^{\bar{C}} > 0$, which ensures that $\exists k > 1$ such that (110) holds. Therefore, if the initial length L is chosen to satisfy (94), then there exists a positive constant $\sigma > 0$ such that the following estimate holds

$$\dot{V} \leq -\sigma V, \tag{111}$$

which completes the proof of Theorem 1.

CONCLUSION

This paper proposes the velocity control of a leading autonomous vehicle to stabilize the oscillations of congested traffic behind it. The mathematical model is based on ARZ traffic model with moving boundaries. We propose the boundary feedback control law after the change of coordinates and state variables. The designed controller is proven to guarantee the physical properties of traffic in the moving domain behind the autonomous vehicle, which ensures the model validity. The closed-loop system is shown to be exponentially stable via Lyapunov analysis. Future work will be focused on the simulation study to illustrate the performance of our control design. Adaptive control design needs to be developed when some parameters in the model, e.g., relaxation time, are unknown.

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